

ON MULTICOLOR RAMSEY NUMBER FOR 3-PATHS OF LENGTH THREE

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ABSTRACT. We show that if we color the hyperedges of the complete 3-uniform complete graph on $2n + \sqrt{18n + 1} + 2$ vertices with n colors, then one of the color classes contains a loose path of length three.

Let P denote the 3-uniform path of length three by which we mean the only 3-uniform connected hypergraph on seven vertices with the degree sequence $(2, 2, 1, 1, 1, 1, 1)$. By $R(P; n)$ we denote the multicolored Ramsey number for P defined as the smallest number N such that each coloring of the hyperedges of the complete 3-uniform hypergraph $K_N^{(3)}$ with n colors leads to a monochromatic copy of P . It is easy to check that $R(P; n) \geq n + 6$ (see, for instance, [2, 5]), and it is believed that in fact the equality holds, i.e.

$$R(P; n) = n + 6.$$

Gyárfás and Raeisi [2] proved, among many other results, that $R(P; 2) = 8$. Their theorem was extended by Omidi and Shahsiah [9] to loose paths of arbitrary lengths, but still only for the case of two colors. On the other hand, in a series of papers [5, 7, 12, 10] it was verified that $R(P; n) = n + 6$ for all $3 \leq n \leq 10$.

Note that from the fact that for $N \geq 8$ the largest P -free 3-uniform hypergraph on N vertices contains at most $\binom{N-1}{2}$ edges (see [6]), it follows that for $n \geq 7$ we have

$$R(P; n) \leq 3n + 1.$$

Our main goal is to improve the above bound.

Theorem. $R(P; n) \leq 2n + \sqrt{18n + 1} + 2$.

Let C denote the (loose) 3-uniform 3-cycle, i.e. the only 3-uniform linear hypergraph with six vertices and three hyperedges. Furthermore, let F be the 3-uniform hypergraph on vertices v_1, v_2, v_3, v_4, v_5 such that the first four of these vertices span in F the clique, and v_5 is contained

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in the following three hyperedges: $v_1v_2v_5$, $v_2v_3v_5$, and $v_3v_4v_5$. The following fact will be crucial for our argument.

Lemma. *Let H be a 3-uniform P -free hypergraph on $n \geq 5$ vertices. Then we can delete from H fewer than $3n$ hyperedges in such a way that the resulting hypergraph contains no copies of C and F .*

Proof. Let us first consider components containing C . Jackowska, Polcyn and Ruciński [7] showed that each such component of H on n_i vertices has at most $3n_i - 8 < 3n_i$ hyperedges, provided $n_i \geq 7$. Furthermore, from the complete 3-hypergraph on $n_i = 6$ vertices it is enough to delete $10 < 3n_i$ edges to get a star, which clearly contains no copies of C and F . Hence, to get rid of all copies of C (and F in components containing C) it is enough to remove at most $3n'$ of hyperedges from components containing them, where n' denotes the number of vertices in these components combined. Now let us consider components containing F but not C . It is easy to check by a direct inspection that any hyperedge e which shares with F just one vertex would create a copy of P . Moreover, any hyperedge e' which shares with F two vertices would create a copy of C . Consequently, each copy of F in a P -free, C -free 3-uniform hypergraph H' is a component. Note that each such component has at most $\binom{5}{3} = 10$ edges and we can destroy F by removing from it $4 < 3 \cdot 5$ hyperedges. Thus, one can delete from H fewer than $3n$ hyperedges to destroy all copies of C and F . \square

Proof of Theorem. Consider a coloring of the edges of the complete 3-uniform hypergraph on $2n + m$ vertices with n colors. Assume that no color class contains a copy of P . Then, by Lemma, we can mark as ‘blank’ fewer than $r = 3(2n + m)n$ hyperedges of the graph in such a way that when we ignore blank hyperedges the coloring contains no monochromatic copies of C and F .

Let us color a pair of vertices vw with a color s , $s = 1, 2, \dots, n$, if there exist at least three hyperedges of color s , $s = 1, 2, \dots, n$, which contain this pair. If there are many such colors we choose any of them; if there are none we leave vw uncolored. Note that every uncolored pair must be contained in at least $m - 2$ blank hyperedges. Consequently, fewer than $3r/(m - 2)$ pairs remains uncolored. But then there exists a color t , $t = 1, 2, \dots, n$, such that there are more than

$$\left[\binom{2n + m}{2} - \frac{3r}{m - 2} \right] / n = 2n + m + (2n + m) \left[\frac{m - 1}{2n} - \frac{9}{m - 2} \right]$$

pairs colored with t . If $m \geq \sqrt{18n + 1} + 2$, then

$$\frac{m - 1}{2n} - \frac{9}{m - 2} > 0,$$

and the graph G_t spanned by these pairs has more edges than vertices. But it means that G_t contains a path of length 3, i.e. there are vertices

v_1, v_2, v_3, v_4 and a color t such that each of the three pairs v_1v_2, v_2v_3, v_3v_4 is contained in at least three different hyperedges colored with t . We shall show that it leads to a contradiction.

Indeed, let H_t be a hypergraph spanned by hyperedges colored with the t th color. Observe first that since v_2v_3 is contained in three different hyperedges of H_t there must be one which is different from $v_1v_2v_3$ and $v_2v_3v_4$; let us call it $v_2v_3v_5$ where $v_5 \neq v_1, v_2, v_3, v_4$. Furthermore, v_1v_2 must be contained in a hyperedge v_1v_2w of H_t where $w \neq v_3, v_5$, while v_3v_4 is contained in some v_3v_4u , where $u \neq v_2, v_5$. Note now that if $w \neq v_4$ and $u \neq w, v_1$, then H_t contains a copy of P which contradicts the fact that it is P -free. The case $w = u \neq v_1, v_4$ would lead to a cycle C , as well as the cases $w = v_4, u \neq v_1$, and $u = v_1, w \neq v_4$. Finally, if the only possible choices for w and u are $w = v_4$ and $u = v_1$, then the vertices v_1, \dots, v_5 span a copy of F , so we arrive at the contradiction again. \square

Remark The bound $3n$ given in Lemma is rather crude. Using the fact that each component of H on n_i vertices containing C and two disjoint edges has at most $n_i + 5$ hyperedges, provided $n_i \geq 7$ (see [10]) and by careful analysis of 3-uniform intersecting families (see [1, 3, 4, 8, 11]) one can improve it by a constant factor and, consequently, improve by a constant factor the second order term in the estimate for $R(P; n)$.

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